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Operator representations of frames: boundedness, duality, and stability.

Ole Christensen, Marzieh Hasannasab

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Abstract

The purpose of the paper is to analyze frames $\{f_k\}_{k \in \mathbb{Z}}$ having the form $\{T^k f_0\}_{k \in \mathbb{Z}}$ for some linear operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$. A key result characterizes boundedness of the operator T in terms of shift-invariance of a certain sequence space. One of the consequences is a characterization of the case where the representation $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ can be achieved for an operator T that has an extension to a bounded bijective operator $\tilde{T} : \mathcal{H} \rightarrow \mathcal{H}$. In this case we also characterize all the dual frames that are representable in terms of iterations of an operator V ; in particular we prove that the only possible operator is $V = (\tilde{T}^*)^{-1}$. Finally, we consider stability of the representation $\{T^k f_0\}_{k \in \mathbb{Z}}$; rather surprisingly, it turns out that the possibility to represent a frame on this form is sensitive towards some of the classical perturbation conditions in frame theory. Various ways of avoiding this problem will be discussed. Throughout the paper the results will be connected with the operators and function systems appearing in applied harmonic analysis, as well as with general group representations.

1 Introduction

In this paper we consider frames $\{f_k\}_{k \in \mathbb{Z}}$ in a Hilbert space \mathcal{H} arising via iterated action of a linear operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$, i.e., on the form

$$\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}. \quad (1.1)$$

We say that the frame $\{f_k\}_{k \in \mathbb{Z}}$ is *represented* via the operator T . The motivation to consider frames of this form comes from several directions:

- The (Fourier) orthonormal basis $\{e^{2\pi i k x}\}_{k \in \mathbb{Z}}$ for $L^2(0, 1)$ has the form (1.1), where $(Tf)(x) = e^{2\pi i x} f(x)$ and $f = \chi_{[0,1]}$.
- Single-generated shift-invariant systems (Example 1.1) and Gabor systems (Example 2.7) have the form (1.1).
- A group representation acting on a cyclic (sub)group indexed by \mathbb{Z} (Example 1.2) leads to a system of vectors on the form (1.1).

The idea of representing frames on the form (1.1) is also closely related with dynamical sampling, see, e.g., [2]. However, the indexing of a frame in the context of dynamical sampling is different from the one used in (1.1), and we will show that a re-indexing might change the properties of the operator T drastically.

In Section 2 we first classify the frames having the form $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$, where T is a linear (not necessarily bounded) operator defined on $\text{span}\{f_k\}_{k \in \mathbb{Z}}$. One of the main results characterizes the frames that can be represented in terms of a bounded operator T , in terms of shift-invariance of a certain subspace of $\ell^2(\mathbb{Z})$. Various consequences of this result are derived, e.g., that if an overcomplete frame with finite excess has a representation of the form (1.1), then T is necessarily unbounded.

Section 3 deals with the properties of the dual frames associated with a frame on the form $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$. For the important case where T is bounded and bijective, we characterize the dual frames that can be represented in terms of a bounded operator V ; in particular, we show that the only possibility of the representing operator for the dual frame is $V = (T^*)^{-1}$.

In Section 4 we consider stability of a representation (1.1) under various perturbation conditions. Rather surprisingly, it turns out that a representation of such a type is unstable under the classical perturbation conditions in frame theory, e.g., the Paley-Wiener type conditions [8, 7]. That is, a perturbation of a frame $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ might not be representable in terms of an operator; or, if the operator T is bounded, a perturbation might turn the frame $\{f_k\}_{k \in \mathbb{Z}}$ into a frame that is only representable in terms of an unbounded operator. We prove, however, that under certain restrictions on the perturbation condition, stability and boundedness is preserved. Finally, for frames $\{f_k\}_{k \in \mathbb{Z}}$ that are norm-bounded below we prove that the type of

perturbation condition that is used most frequently in the literature leads to frames $\{g_k\}_{k \in \mathbb{Z}}$ that can be represented via iterations of a *finite* collection of bounded operators.

The paper closes with an appendix, containing some operator theoretic considerations. We show, e.g., that the chosen indexing is important for the properties of the operator representing a given frame.

In the rest of the introduction, we will collect some definitions and standard results from frame theory. A sequence $\{f_k\}_{k=1}^\infty$ in a Hilbert space \mathcal{H} is a *frame* for \mathcal{H} if there exist constants $A, B > 0$ such that $A\|f\|^2 \leq \sum_{k=1}^\infty |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \forall f \in \mathcal{H}$. The sequence $\{f_k\}_{k=1}^\infty$ is a Bessel sequence if at least the upper frame condition holds. Also, $\{f_k\}_{k=1}^\infty$ is called a *Riesz sequence* if there exist constants $A, B > 0$ such that $A \sum |c_k|^2 \leq \|\sum c_k f_k\|^2 \leq B \sum |c_k|^2$ for all finite scalar sequences $c_1, \dots, c_N, N \in \mathbb{N}$.

If $\{f_k\}_{k \in \mathbb{Z}}$ is a Bessel sequence, the *synthesis operator* is defined by

$$U : \ell^2(\mathbb{Z}) \rightarrow \mathcal{H}, U\{c_k\}_{k \in \mathbb{Z}} := \sum_{k \in \mathbb{Z}} c_k f_k; \quad (1.2)$$

it is well known that U is well-defined and bounded. A central role will be played by the kernel of the operator U , i.e., the subset of $\ell^2(\mathbb{Z})$ given by

$$N_U = \left\{ \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \mid \sum_{k \in \mathbb{Z}} c_k f_k = 0 \right\}. \quad (1.3)$$

The *excess* of a frame is the number of elements that can be removed yet leaving a frame. It is well-known that the excess equals $\dim(N_U)$; see [4].

Given a Bessel sequence $\{f_k\}_{k \in \mathbb{Z}}$, the *frame operator* $S : \mathcal{H} \rightarrow \mathcal{H}$ is defined by $S := UU^*$. For a frame $\{f_k\}_{k \in \mathbb{Z}}$, the frame operator is invertible and $f = \sum_{k \in \mathbb{Z}} \langle f, S^{-1} f_k \rangle f_k, \forall f \in \mathcal{H}$. The sequence $\{S^{-1} f_k\}_{k \in \mathbb{Z}}$ is also a frame; it is called the *canonical dual frame*.

For a frame $\{f_k\}_{k \in \mathbb{Z}}$ that is not a Riesz basis, it is known that there exists infinitely many *dual frames*, i.e., frames $\{g_k\}_{k \in \mathbb{Z}}$ such that

$$f = \sum_{k \in \mathbb{Z}} \langle f, g_k \rangle f_k, \forall f \in \mathcal{H}.$$

The class of dual frames have been characterized by Li [13].

Throughout the paper we will illustrate the results with applications to frames appearing in applied harmonic analysis, e.g., shift-invariant systems

and Gabor systems. First, for $a \in \mathbb{R}$, define the *translation operator* T_a acting on $L^2(\mathbb{R})$ by $T_a f(x) := f(x - a)$ and the *modulation operator* E_a by $E_a f(x) := e^{2\pi i a x} f(x)$. Both operators are unitary. Furthermore, defining the Fourier transform of $f \in L^1(\mathbb{R})$ by $\widehat{f}(\gamma) = \mathcal{F}f(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x} dx$ and extend it in the standard way to a unitary operator on $L^2(\mathbb{R})$, we have $\mathcal{F}T_a = E_{-a}\mathcal{F}$. The following example will inspire us throughout the paper.

Example 1.1 Consider a function $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$. Then the *shift-invariant system* $\{T_k \varphi\}_{k \in \mathbb{Z}}$ is linearly independent. Letting $\Phi(\gamma) := \sum_{k \in \mathbb{Z}} |\widehat{\varphi}(\gamma + k)|^2$, it was proved in [6] (or see Theorem 9.2.5 in [9]) that $\{T_k \varphi\}_{k \in \mathbb{Z}}$ is a frame sequence if and only if there exist $A, B > 0$ such that $A \leq \Phi(\gamma) \leq B$, a.e. $\gamma \in [0, 1] \setminus N$, where $N =: \{\gamma \in [0, 1] \mid \sum_{k \in \mathbb{Z}} |\widehat{\varphi}(\gamma + k)|^2 = 0\}$. Furthermore, the special case where $\{T_k \varphi\}_{k \in \mathbb{Z}}$ is a Riesz sequence corresponds to the case where the set N has measure zero. Note that regardless of the frame properties of the shift-invariant system, we can write it on the form $\{T_k \varphi\}_{k \in \mathbb{Z}} = \{(T_1)^k \varphi\}_{k \in \mathbb{Z}}$, i.e., as the iterated system arising by letting the powers of the bounded operator T_1 act on the function φ . \square

More generally, iterated systems $\{T^k f_0\}_{k \in \mathbb{Z}}$ naturally shows up in the context of group representations. This topic is well connected with frame theory; see, e.g., the paper [5] and the references therein.

Example 1.2 Let G denote a locally compact group, and π a group representation of G on a Hilbert space \mathcal{H} ; that is π is a mapping from G into the space of bounded invertible operators on \mathcal{H} , satisfying that $\pi(xy) = \pi(x)\pi(y)$ for all $x, y \in G$. Now, fix some $x_0 \in G$. Considering the cyclic subgroup of G generated by the element x_0 , i.e., the set $\{x_k\}_{k \in \mathbb{Z}} = \{x_0^k\}_{k \in \mathbb{Z}} \subset G$, the group representation acting on a fixed $f_0 \in \mathcal{H}$ generates the family of vectors $\{\pi(x_0^k) f_0\}_{k \in \mathbb{Z}} = \{[\pi(x_0)]^k f_0\}_{k \in \mathbb{Z}}$. This system has the structure (1.1) with $T = \pi(x_0)$. Note that Example 1.1 is a special case of this; indeed, the left-regular representation of the group \mathbb{R} with the composition " + " on $L^2(\mathbb{R})$ is precisely $(\pi(x)f)(t) = f(t - x)$, $f \in L^2(\mathbb{R})$, $t, x \in \mathbb{R}$. The general setting of group representations covers this example and its discrete variant in $\ell^2(\mathbb{Z})$ (whose frame properties are analyzed in [12]) in a unifying way.

Note that the structure of systems arising from a group representation is very rigid: arbitrary small perturbations might destroy the special structure, so it is important that such cases can still be handled within the frame work of the more general systems (1.1). \square

2 The representation $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$

In this section we want to consider representation of a frame $\{f_k\}_{k \in \mathbb{Z}}$ on the form $\{T^k f_0\}_{k \in \mathbb{Z}}$ for some linear operator T defined on an appropriate subspace of \mathcal{H} . The starting point must be a clarification of the exact meaning of $\{T^k f_0\}_{k \in \mathbb{Z}}$. For $k \geq 0$, this is clear. For $k = -1$ we will require that T is invertible as a map from $\text{span}\{f_k\}_{k \in \mathbb{Z}}$ into itself. This guarantees that $T^{-1}f_0$ is well-defined, and hence also $T^k f_0 = (T^{-1})^{-k} f_0$ is well-defined for $k = -2, -3, \dots$. In the following result, we characterize the availability of the representation $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$. The proof is a modification of the corresponding result for sequences indexed by \mathbb{N} , so we only sketch it.

Proposition 2.1 *Consider a frame sequence $\{f_k\}_{k \in \mathbb{Z}}$ in a Hilbert space \mathcal{H} which spans an infinite dimensional subspace. The following are equivalent:*

- (i) $\{f_k\}_{k \in \mathbb{Z}}$ is linearly independent.
- (ii) The map $Tf_k := f_{k+1}$ is well-defined, and extends to a linear and invertible operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$.

In the affirmative case, $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$.

Proof. The proof that (i) \Rightarrow (ii) is exact the same as for sequences indexed by \mathbb{N} , see [10]. Now assume that (ii) holds. It is easy to see that $f_k \neq 0$, for all $k \in \mathbb{Z}$. Now in order to reach a contradiction, assume that $\sum_{k=M}^N c_k f_k = 0$. For some coefficient c_k , $k = M, \dots, N$ not all of which are zero. We can choose $M, N \in \mathbb{Z}$ such that $c_M \neq 0$, $c_N \neq 0$. Then, the same proof as in [10] shows that the vector space $V := \text{span}\{f_k\}_{k=M}^N$ is invariant under the action of T . Now, a similar calculation shows that V is invariant under the action of T^{-1} . Thus $\text{span}\{f_k\}_{k \in \mathbb{Z}} = \text{span}\{T^k f_0\}_{k \in \mathbb{Z}} = \text{span}\{f_k\}_{k=M}^N = V$, which is a contradiction because $\text{span}\{f_k\}_{k \in \mathbb{Z}}$ is assumed to be infinite-dimensional. Thus $\{f_k\}_{k \in \mathbb{Z}}$ is linearly independent, as desired. \square

If $\{f_k\}_{k \in \mathbb{Z}}$ is a frame sequence and the operator T in Proposition 2.1 is bounded, it has a unique extension to a bounded operator $\tilde{T} : \overline{\text{span}}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \overline{\text{span}}\{f_k\}_{k \in \mathbb{Z}}$, given by

$$\tilde{T} \sum_{k \in \mathbb{Z}} c_k f_k = \sum_{k \in \mathbb{Z}} c_k f_{k+1}, \quad \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

We first state a necessary condition in order for a frame $\{f_k\}_{k \in \mathbb{Z}}$ to have a representation on the form $\{T^k f_0\}_{k \in \mathbb{Z}}$ for a given bounded operator T .

Proposition 2.2 Consider a frame on the form $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ for some bounded linear operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$. Then the following hold:

(i) $\|T\| \geq 1$.

(ii) If $T^{-1} : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$ is bounded, then $\|T^{-1}\| \geq 1$.

Proof. Let A, B denote some frame bounds and fix any $n \in \mathbb{N}$. Using the frame inequalities for any $f \neq 0$, we have

$$\begin{aligned} A\|f\|^2 &\leq \sum_{k \in \mathbb{Z}} |\langle f, T^k f_0 \rangle|^2 = \sum_{k \in \mathbb{Z}} |\langle (T^n)^* f, T^{k-n} f_0 \rangle|^2 \\ &= \sum_{k \in \mathbb{Z}} |\langle (T^n)^* f, T^k f_0 \rangle|^2 \leq B\|(T^n)^* f\|^2 \leq B\|T\|^{2n}\|f\|^2. \end{aligned} \quad (2.1)$$

Therefore $A \leq B\|T\|^{2n}$ for all $n \in \mathbb{N}$, which implies that $\|T\| \geq 1$. The result in (ii) follows by replacing T by T^{-1} and noticing that these two operators represent the same frame. \square

Assuming that a frame $\{f_k\}_{k \in \mathbb{Z}}$ has a representation on the form $\{T^k f_0\}_{k \in \mathbb{Z}}$, we will now characterize boundedness of the operator T in terms of the kernel of the synthesis operator, see (1.3); in particular, this leads to a characterization of the case where the operator T has an extension to a bounded bijective operator on \mathcal{H} .

For this purpose we need the analogue of the translation operator, acting on the sequence space $\ell^2(\mathbb{Z})$. Define the *right-shift operator* on $\ell^2(\mathbb{Z})$ by

$$\mathcal{T} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z}), \mathcal{T}\{c_k\}_{k \in \mathbb{Z}} = \{c_{k-1}\}_{k \in \mathbb{Z}}. \quad (2.2)$$

Clearly \mathcal{T} is a unitary operator on $\ell^2(\mathbb{Z})$. We say that a subspace $V \subseteq \ell^2(\mathbb{Z})$ is *invariant under right-shifts* (respectively, left-shifts) if $\mathcal{T}(V) \subseteq V$ (respectively, if $\mathcal{T}^{-1}(V) \subseteq V$).

Theorem 2.3 Consider a frame having the form $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ for some linear operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$, and let $A, B > 0$ denote some frame bounds. Then the following hold:

(i) The operator T is bounded if and only if the kernel N_U of the synthesis operator U is invariant under right-shifts; in the affirmative case,

$$1 \leq \|T\| \leq \sqrt{BA^{-1}}.$$

(ii) The operator $T^{-1} : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$ is bounded if and only if N_U is invariant under left-shifts; in the affirmative case,

$$1 \leq \|T^{-1}\| \leq \sqrt{BA^{-1}}.$$

(iii) Assume that N_U is invariant under right and left-shifts. Then the operator T has an extension to a bounded bijective operator $\tilde{T} : \mathcal{H} \rightarrow \mathcal{H}$.

Proof. In order to prove (i), assume that $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ for a bounded operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$. Then T can be extended to a bounded operator $\tilde{T} : \mathcal{H} \rightarrow \mathcal{H}$. For any $\{c_k\}_{k \in \mathbb{Z}} \in N_U$, we have

$$UT\{c_k\}_{k \in \mathbb{Z}} = \sum_{k=-\infty}^{\infty} c_{k-1}f_k = \sum_{k=-\infty}^{\infty} c_k f_{k+1} = \tilde{T} \sum_{k=-\infty}^{\infty} c_k f_k = T0 = 0.$$

Therefore $\mathcal{T}(N_U) \subseteq N_U$, as claimed.

Conversely, assume that N_U is invariant under right-shifts. Assume that $f \in \text{span}\{f_k\}_{k \in \mathbb{Z}}$, i.e., $f = \sum_{k=M}^N c_k f_k$ for some $M, N \in \mathbb{Z}, c_k \in \mathbb{C}$. One can consider $\{c_k\}_{k=M}^N$ as a sequence $\{c_k\}_{k \in \mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ where $c_k = 0$ for $k > N$ and $k < M$. Thus we can write $\{c_k\}_{k \in \mathbb{Z}} = \{d_k\}_{k \in \mathbb{Z}} + \{r_k\}_{k \in \mathbb{Z}}$, where $\{d_k\}_{k \in \mathbb{Z}} \in N_U$ and $\{r_k\}_{k \in \mathbb{Z}} \in N_U^\perp$. Since N_U is invariant under right-shifts, we have $\sum_{k \in \mathbb{Z}} d_k f_{k+1} = 0$. Using the splitting of $\{c_k\}_{k \in \mathbb{Z}}$ and that $\{f_k\}_{k \in \mathbb{Z}}$ is a Bessel sequence, we get that

$$\begin{aligned} \|Tf\|^2 &= \left\| \sum_{k=M}^N c_k f_{k+1} \right\|^2 = \left\| \sum_{k=-\infty}^{\infty} (d_k + r_k) f_{k+1} \right\|^2 = \left\| \sum_{k=-\infty}^{\infty} r_k f_{k+1} \right\|^2 \\ &\leq B \sum_{k=-\infty}^{\infty} |r_k|^2. \end{aligned}$$

Recall (see Lemma 5.5.5 in [9]) that since $\{f_k\}_{k \in \mathbb{Z}}$ is a frame with lower bound A , we have $A \sum_{k \in \mathbb{Z}} |c_k|^2 \leq \|U\{c_k\}_{k \in \mathbb{Z}}\|^2$, $\forall \{c_k\}_{k \in \mathbb{Z}} \in N_U^\perp$. It follows that

$$\|Tf\|^2 \leq \frac{B}{A} \left\| \sum_{k \in \mathbb{Z}} r_k f_k \right\|^2 = \frac{B}{A} \left\| \sum_{k \in \mathbb{Z}} c_k f_k \right\|^2 = \frac{B}{A} \|f\|^2,$$

i.e., T is bounded as desired. The above calculations also confirm the claimed upper bound on the norm of T . The lower bound in the estimate in (i) was

proved in Proposition 2.2; this completes the proof of (i). The result (ii) is a consequence of (i). Indeed, since $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$, we can write $\{f_{-k}\}_{k \in \mathbb{Z}} = \{(T^{-1})^k f_0\}_{k \in \mathbb{Z}}$. Denoting the synthesis operator for $\{f_{-k}\}_{k \in \mathbb{Z}}$ by V , Theorem 2.3 shows that T^{-1} is bounded if and only if the kernel N_V is right-shifts invariant. It is easy to see that $\{c_k\}_{k \in \mathbb{Z}} \in N_U$ if and only if $\{c_{-k}\}_{k \in \mathbb{Z}} \in N_V$. Hence the left-shifts invariance of N_U is equivalent with the right-shift invariance of N_V .

For the proof of (iii), if N_U is invariant under right and left-shifts, then the operators $T, T^{-1} : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$ are bounded. Hence they can be extended to bounded operators $\widetilde{T}, \widetilde{T}^{-1}$ on \mathcal{H} . Since

$$T^{-1}Tf = TT^{-1}f = f, \quad f \in \text{span}\{f_k\}_{k \in \mathbb{Z}},$$

it follows that $\widetilde{T}\widetilde{T}^{-1} = \widetilde{T}^{-1}\widetilde{T} = I$, i.e., \widetilde{T} is invertible on \mathcal{H} . \square

Throughout the paper it will be crucial to distinguish carefully between a bounded operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$ and its extension $\widetilde{T} : \mathcal{H} \rightarrow \mathcal{H}$. Indeed, our setup implies that T is invertible, but the extension to an operator on \mathcal{H} might no longer be injective (for the convenience of the interested reader we include such an example in the Appendix).

Note that the biimplications in Theorem 2.3 uses the full strength of the frame assumption. Indeed, one can construct examples of sequences $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ satisfying only the upper frame condition (resp. the lower frame condition), and such that T is unbounded while the kernel N_U is invariant under right-shifts.

Let us demonstrate the power of Theorem 2.3 by some consequences and examples; another application will be given in Proposition 4.1. Let us first consider the special case of a Riesz sequence.

Corollary 2.4 *Any Riesz sequence $\{f_k\}_{k \in \mathbb{Z}}$ has a representation $\{T^k f_0\}_{k \in \mathbb{Z}}$ for a bounded and bijective operator $T : \overline{\text{span}}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \overline{\text{span}}\{f_k\}_{k \in \mathbb{Z}}$.*

Corollary 2.4 follows immediately from Theorem 2.3 and the fact that the synthesis operator for a Riesz sequence is injective. We therefore now turn to the setting of an overcomplete frame.

Corollary 2.5 *Consider an overcomplete frame on the form $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$. If $T \in B(\mathcal{H})$, then $\dim(N_U) = \infty$.*

Proof. If there is a nonzero element $c = \{c_k\}_{k \in \mathbb{Z}}$ in N_U , then by Theorem 2.3, the boundedness of T implies that $\mathcal{T}^j c \in N_U$ for all $j \in \mathbb{N}$. We will now show that the sequence $\{\mathcal{T}^j c\}_{j=1}^\infty$ is linearly independent; this implies that N_U is infinite-dimensional and concludes the proof. Now consider the operator $\mathcal{F} : \ell^2(\mathbb{Z}) \mapsto L^2[0, 1]$, $\mathcal{F}c = \sum_{k \in \mathbb{Z}} c_k e_k$, where $e_k(x) = e^{-2\pi i k x}$. The operator \mathcal{F} is unitary, and $\mathcal{F}\mathcal{T}c = e_1 \mathcal{F}c$. Now assume that for some $N \in \mathbb{N}$ and $d_1, d_2, \dots, d_N \in \mathbb{C}$, we have $\sum_{j=1}^N d_j \mathcal{T}^j c = 0$. Let $\varphi = \mathcal{F}c$. Then

$$0 = \mathcal{F}\left(\sum_{j=1}^N d_j \mathcal{T}^j c\right) = \sum_{j=1}^N d_j e_j \mathcal{F}c = \left(\sum_{j=1}^N d_j e_j\right) \varphi.$$

This means that $(\sum_{j=1}^N d_j e_j(x))\varphi(x) = 0$, for *a.e.* $x \in [0, 1]$. Since $\varphi \neq 0$, the support of φ has positive measure. Thus we have $\sum_{j=1}^N d_j e_j(x) = 0$ for all $x \in \text{supp } \varphi$ which implies that $d_j = 0$ for $j = 1, 2, \dots, N$. Thus the sequence $\{\mathcal{T}^j c\}_{j=1}^\infty$ is linearly independent, as desired. \square

Corollary 2.5 leads to a general result about arbitrary group representations and the operators generated by cyclic subgroups indexed by \mathbb{Z} :

Corollary 2.6 *Let G denote a locally compact group, and π a group representation of G on a Hilbert space \mathcal{H} . Given any $x_0 \in G$ and any $f_0 \in \mathcal{H}$, and assume that the family $\{\pi(x_0^k) f_0\}_{k \in \mathbb{Z}} = \{\pi(x_0)^k f_0\}_{k \in \mathbb{Z}}$ is a frame sequence. Then either the family is a Riesz sequence, or it has infinite excess.*

The result in Corollary 2.6 is known in certain special cases, e.g., for the case of a shift-invariant system considered in Example 1.1.

Note that the opposite implication in Corollary 2.5 does not hold; that is, the operator T is not necessarily bounded even if $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ is an overcomplete frame and $\dim(N_U) = \infty$. This is demonstrated by the following example.

Example 2.7 A collection of functions in $L^2(\mathbb{R})$ of the form $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ for some $a, b > 0$ and some $g \in L^2(\mathbb{R})$ is called a *Gabor system*. It is known that if $g \neq 0$, then the Gabor system $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is automatically linearly independent, see [14, 11]; thus it can be represented on the form $\{T^k f_0\}_{k \in \mathbb{Z}}$. Now, consider the Gabor frame $\{E_{m/3} T_n \chi_{[0,1]}\}_{m,n \in \mathbb{Z}}$, which is the union of the three orthonormal bases $\{E_{k/3} E_m T_n \chi_{[0,1]}\}_{m,n \in \mathbb{Z}}$, $k = 0, 1, 2$. The Gabor frame $\{E_{m/3} T_n \chi_{[0,1]}\}_{m,n \in \mathbb{Z}}$ is linearly independent and has infinite excess; in particular $\dim(N_U) = \infty$. Re-order the frame as $\{f_k\}_{k \in \mathbb{Z}}$

in such a way that the elements $\{f_{2k+1}\}_{k \in \mathbb{Z}}$ corresponds to the orthonormal basis $\{E_m T_n \chi_{[0,1]}\}_{m,n \in \mathbb{Z}}$. By construction, the elements $\{f_{2k}\}_{k \in \mathbb{Z}}$ now forms an overcomplete frame. By Proposition 2.1, there is an operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$ such that $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$. Since the subsequence $\{f_{2k}\}_{k \in \mathbb{Z}}$ is an overcomplete frame, there is a non-zero sequence $\{c_{2k}\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that $\sum_{k \in \mathbb{Z}} c_{2k} f_{2k} = 0$. Defining $c_k = 0$ for $k \in 2\mathbb{Z} + 1$, we have $\sum_{k \in \mathbb{Z}} c_k f_k = \sum_{k \in \mathbb{Z}} c_{2k} f_{2k} = 0$. On the other hand, since $\{f_{2k+1}\}_{k \in \mathbb{Z}}$ is a Riesz basis and $\{c_k\}_{k \in \mathbb{Z}}$ is non-zero, $\sum_{k \in \mathbb{Z}} c_k f_{k+1} = \sum_{k \in \mathbb{Z}} c_{2k} f_{2k+1} \neq 0$. This shows that N_U is not invariant under right-shifts; thus, T is unbounded by Theorem 2.3. \square

If $\{f_k\}_{k \in \mathbb{Z}}$ is a Riesz basis on the form $\{T^k f_0\}_{k \in \mathbb{Z}}$, then the extension of the bounded operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$ to \mathcal{H} is injective. On the other hand, if a given frame has the form $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ for a bounded and injective operator on \mathcal{H} , we can not conclude that $\{f_k\}_{k \in \mathbb{Z}}$ is a Riesz basis:

Example 2.8 Using the characterization in Example 1.1, it is easy to construct an overcomplete frame sequence $\{T_k \varphi\}_{k \in \mathbb{Z}} = \{(T_1)^k \varphi\}_{k \in \mathbb{Z}}$ in $L^2(\mathbb{R})$; in other words, letting $\mathcal{H} := \overline{\text{span}}\{T_k \varphi\}_{k \in \mathbb{Z}}$ the sequence $\{T_k \varphi\}_{k \in \mathbb{Z}}$ is an overcomplete frame for \mathcal{H} . Clearly T_1 is a bounded and injective operator on \mathcal{H} , but by construction $\{T_k \varphi\}_{k \in \mathbb{Z}}$ is not a Riesz basis for \mathcal{H} . \square

Corollary 2.9 *Consider a tight frame having a representation $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ for some invertible operator $T \in B(\mathcal{H})$. Then T is an isometry.*

Proof. Since the frame bounds are $A = B$, using Theorem 2.3, we have $\|T\| = \|T^{-1}\| = 1$. Therefore $\|f\| = \|T^{-1} T f\| \leq \|T f\| \leq \|f\|$, which implies that T is in isometry. \square

3 Duality

In this section we will analyze certain aspects of the duality theory for a frame having the form $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ for some bounded linear and invertible operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$. In particular we will identify a class of dual frames (including the canonical dual frame) that is

also given by iteration of a bounded operator. On the other hand, we also give an example of a frame for which not all dual frames have this form.

In the entire section we denote the synthesis operator by U ; then the frame operator is $S = UU^*$. We first prove that the synthesis operator U is an intertwining operator for the right-shift operator \mathcal{T} on $\ell^2(\mathbb{Z})$ and the operator T , as well as an immediate consequence for the frame operator. Let $c_{00} \subset \ell^2(\mathbb{Z})$ denote the subspace consisting of finite sequences.

Lemma 3.1 *Consider a Bessel sequence having the form $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ for a linear operator $T : \text{span}\{f_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{f_k\}_{k \in \mathbb{Z}}$. Then $TU = U\mathcal{T}$ on c_{00} . Assuming that T has an extension to a bounded operator $\tilde{T} : \mathcal{H} \rightarrow \mathcal{H}$, the following hold:*

- (i) $\tilde{T}U = U\mathcal{T}$ on $\ell^2(\mathbb{Z})$.
- (ii) If $\{T^k f_0\}_{k \in \mathbb{Z}}$ is a frame and \tilde{T} is invertible, then $\tilde{T}S = S(\tilde{T}^*)^{-1}$; in particular, $S\tilde{T} = \tilde{T}S$ if and only if \tilde{T} is unitary.

Proof. For $\{c_k\}_{k \in \mathbb{Z}} \in c_{00}$, there is an $N \in \mathbb{N}$ such that $c_k = 0$ for $|k| \geq N$. Therefore

$$\begin{aligned} TU\{c_k\}_{k \in \mathbb{Z}} &= T \sum_{k=-N}^N c_k f_k = \sum_{k=-N}^N c_k f_{k+1} = \sum_{k=-N+1}^{N+1} c_{k-1} f_k \\ &= U\{c_{k-1}\}_{k \in \mathbb{Z}} = U\mathcal{T}\{c_k\}_{k \in \mathbb{Z}}. \end{aligned} \quad (3.1)$$

In the case that \tilde{T} is bounded, the equality (3.1) holds on $\ell^2(\mathbb{Z})$ because c_{00} is dense in $\ell^2(\mathbb{Z})$; this proves (i). For the proof of (ii), using (3.1) and that $S = UU^*$,

$$\tilde{T}S\tilde{T}^* = \tilde{T}UU^*\tilde{T}^* = \tilde{T}U(\tilde{T}U)^* = U\mathcal{T}(U\mathcal{T})^* = U\mathcal{T}\mathcal{T}^*U^* = UU^* = S.$$

Therefore $\tilde{T}S = S(\tilde{T}^*)^{-1}$, as desired. \square

For a frame $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$, the operator $S^{-1}TS$ is invertible considered from $\text{span}\{S^{-1}f_k\}_{k \in \mathbb{Z}}$ into itself, and the canonical dual frame is $\{S^{-1}f_k\}_{k \in \mathbb{Z}} = \{(S^{-1}TS)^k S^{-1}f_0\}_{k \in \mathbb{Z}}$. This was already observed in the finite-dimensional setting in [1]. In the case where T has an extension to a bounded and invertible operator on \mathcal{H} (see the appropriate conditions in Theorem 2.3), we will now derive an alternative description of the canonical dual frame, directly in terms of the operator T and its adjoint. Since the rest of the results

in the current section will use the same assumptions on the operator T , we will drop the distinction between the operator T and \tilde{T} , and simply denote the operator by T .

Proposition 3.2 *Consider a frame $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$, where $T \in B(\mathcal{H})$ is invertible. Let $\tilde{f}_0 = S^{-1} f_0$. Then $\{S^{-1} f_k\}_{k \in \mathbb{Z}} = \{(T^*)^{-k} \tilde{f}_0\}_{k \in \mathbb{Z}}$.*

Proof. Lemma 3.1 (ii) implies that $TS = S(T^*)^{-1}$. Thus $S^{-1}T = (T^*)^{-1}S^{-1}$ and therefore $S^{-1}T^k = (T^*)^{-k}S^{-1}$ for $k \in \mathbb{N}$. We also have that $S^{-1}T^{-1} = T^*S^{-1}$ and thus $S^{-1}T^{-k} = (T^*)^kS^{-1}$ for $k \in \mathbb{N}$. It follows that

$$\{S^{-1}f_k\}_{k \in \mathbb{Z}} = \{S^{-1}T^k f_0\}_{k \in \mathbb{Z}} = \{(T^*)^{-k}S^{-1}f_0\}_{k \in \mathbb{Z}} = \{(T^*)^{-k}\tilde{f}_0\}_{k \in \mathbb{Z}},$$

as desired. \square

Since the translation operators on $L^2(\mathbb{R})$ are unitary, Proposition 3.2 generalizes the well-known result that the canonical dual of a shift-invariant frame $\{T_k \varphi\}_{k \in \mathbb{Z}}$ in $L^2(\mathbb{R})$ has the form $\{T_k \tilde{\varphi}\}_{k \in \mathbb{Z}}$ for some $\tilde{\varphi} \in L^2(\mathbb{R})$.

It is important to notice that Proposition 3.2 only shows that the canonical dual frame has the form of an iterated system. Indeed, the next example exhibits a frame satisfying the conditions in Proposition 3.2 and having a dual frame that is not representable by an operator:

Example 3.3 Let us return to Example 1.1 and consider an overcomplete frame sequence $\{T_k \varphi\}_{k \in \mathbb{Z}}$ in $L^2(\mathbb{R})$. Then there exists an element $T_{k'} \varphi$, $k' \in \mathbb{Z}$, that can be removed from the frame sequence, leaving a frame sequence for the same space; due to the special structure of the frame we can even take $k' = 0$. Letting $\{g_k\}_{k=-\infty}^{-1} \cup \{g_k\}_{k=1}^{\infty}$ denote a dual frame for the resulting frame sequence $\{T_k \varphi\}_{k=-\infty}^{-1} \cup \{T_k \varphi\}_{k=1}^{\infty}$ this implies that the frame $\{T_k \varphi\}_{k \in \mathbb{Z}}$ has the non-canonical dual $\{g_k\}_{k=-\infty}^{-1} \cup \{0\} \cup \{g_k\}_{k=1}^{\infty}$; this family is clearly linearly dependent. Hence, by Theorem 2.1 the system is not representable by an operator. \square

We will now show that despite the obstruction in Example 3.3 we can actually characterize the class of dual frames that arise through iterated actions of a bounded operator. We first show that the only candidate for this operator indeed is the operator $(T^*)^{-1}$ arising in Proposition 3.2. In particular, this shows that for a frame $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ given in terms of a unitary operator T , the dual frames having the form of an iterated operator system must be generated by the same operator.

Lemma 3.4 Consider a frame $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$, where $T \in B(\mathcal{H})$ is invertible. Assume that $\{g_k\}_{k \in \mathbb{Z}} = \{V^k g_0\}_{k \in \mathbb{Z}}$ is a dual frame and that V is bounded. Then $V = (T^*)^{-1}$.

Proof. For any $f \in B(\mathcal{H})$, two applications of the frame decomposition yield that

$$\begin{aligned} f &= \sum_{k \in \mathbb{Z}} \langle f, T^k f_0 \rangle V^k g_0 = V \sum_{k \in \mathbb{Z}} \langle f, T^k f_0 \rangle V^{k-1} g_0 \\ &= V \sum_{k \in \mathbb{Z}} \langle T^* f, T^{k-1} f_0 \rangle V^{k-1} g_0 = V \sum_{k \in \mathbb{Z}} \langle T^* f, T^k f_0 \rangle V^k g_0 = VT^* f. \end{aligned}$$

Therefore $VT^* = I$. Since T is invertible it follows that $V = (T^*)^{-1}$. \square

We will now give the full characterization of dual frames of $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ that are given in terms of iterations of a bounded operator.

Theorem 3.5 Consider a frame $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$, where $T \in B(\mathcal{H})$ is invertible. Then the dual frames given as iterates of a bounded operator are precisely the families of the form $\{(T^*)^{-k} g_0\}_{k \in \mathbb{Z}}$ for which

$$g_0 = S^{-1} f_0 + h_0 - \sum_{j \in \mathbb{Z}} \langle S^{-1} f_0, T^j f_0 \rangle (T^*)^{-j} h_0 \quad (3.2)$$

for some $h_0 \in \mathcal{H}$ such that $\{(T^*)^{-k} h_0\}_{k \in \mathbb{Z}}$ is a Bessel sequence. In particular, this condition is satisfied when h_0 is taken from the dense subspace $\text{span}\{(T^*)^{-k} \tilde{f}_0\}_{k \in \mathbb{Z}}$.

Proof. First, note that by Lemma 3.4 we know that the only operator that might be applicable in the desired representation of the dual frame is $(T^*)^{-1}$. Now, assume that $\{(T^*)^{-k} \varphi\}_{k \in \mathbb{Z}}$ is a dual frame of $\{T^k f_0\}_{k \in \mathbb{Z}}$ for some $\varphi \in \mathcal{H}$. Then $\{(T^*)^{-k} \varphi\}_{k \in \mathbb{Z}}$ is a Bessel sequence, and taking $h_0 := \varphi$ in (3.2) yields that $g_0 = \varphi$. On the other hand, assume that $h_0 \in \mathcal{H}$ is chosen such that the sequence $\{(T^*)^{-k} h_0\}_{k \in \mathbb{Z}}$ is a Bessel sequence, and choose g_0 as in (3.2). Denote the synthesis operator of $\{(T^*)^{-k} h_0\}_{k \in \mathbb{Z}}$ by W . Letting $\{\delta_j\}_{j \in \mathbb{Z}}$ denote the canonical orthonormal basis for $\ell^2(\mathbb{Z})$ and $V := S^{-1}U + W(I - U^*S^{-1}U)$, it follows from [13] (alternatively, see Lemma 6.3.5 and Lemma 6.3.6 in [9]) that the sequence $\{V\delta_j\}_{j \in \mathbb{Z}}$ is a dual frame of $\{T^k f_0\}_{k \in \mathbb{Z}}$. Furthermore, by direct calculation,

$$V\delta_0 = (S^{-1}U + W(I - U^*S^{-1}U))\delta_0 = g_0.$$

We first show that this frame indeed has the form $\{(T^*)^{-k}g_0\}_{k \in \mathbb{Z}}$. We will now show that V is an intertwining operator between \mathcal{T} and $(T^*)^{-1}$. Applying Lemma 3.1 (i) on $\{(T^*)^{-k}h_0\}_{k \in \mathbb{Z}}$, we know that $W\mathcal{T} = (T^*)^{-1}W$ and $TU = U\mathcal{T}$. Also since T is bounded and invertible, Lemma 3.1 (ii) shows that $S^{-1}T = (T^*)^{-1}S^{-1}$. Hence we get

$$\begin{aligned} V\mathcal{T} &= (S^{-1}U + W(I - U^*S^{-1}U))\mathcal{T} \\ &= S^{-1}U\mathcal{T} + W\mathcal{T} - WU^*S^{-1}U\mathcal{T} \\ &= S^{-1}TU + (T^*)^{-1}W - WU^*S^{-1}TU \\ &= (T^*)^{-1}S^{-1}U + (T^*)^{-1}W - WU^*(T^*)^{-1}S^{-1}U. \end{aligned}$$

Similarly to the proof of Lemma 3.1, $T^{-1}U = U\mathcal{T}^{-1}$. Therefore $U^*(T^*)^{-1} = (T^{-1}U)^* = (U\mathcal{T}^{-1})^* = \mathcal{T}U^*$; thus,

$$V\mathcal{T} = (T^*)^{-1}(S^{-1}U + W - WU^*S^{-1}U) = (T^*)^{-1}V.$$

This implies that $V\delta_j = V\mathcal{T}^j\delta_0 = (T^*)^{-j}V\delta_0 = (T^*)^{-j}g_0$, as desired.

Finally, we note that if $h_0 \in \text{span}\{(T^*)^{-k}\tilde{f}_0\}_{k \in \mathbb{Z}}$, the sequence $\{(T^*)^{-k}h_0\}_{k \in \mathbb{Z}}$ is a finite sum of frame sequences and hence a Bessel sequence. \square

In order to apply Proposition 3.2 and Theorem 3.5 we must calculate the adjoint of the operator T arising in the representation $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$. In general this can only be done with specific knowledge of the operator T at hand. An additional condition on the frame $\{f_k\}_{k \in \mathbb{Z}}$ implies that the operator T is unitary, and allows us to find it explicitly in terms of $\{f_k\}_{k \in \mathbb{Z}}$; the result generalizes the observations for shift-invariant systems in Example 1.1, and also applies to some of the other systems obtained via group representations in Example 1.2.

Proposition 3.6 *Consider a frame having the form $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ for some operator $T \in B(\mathcal{H})$. Assume that for a function $\theta : \mathbb{Z} \mapsto \mathbb{R}$, we have $\langle f_j, f_k \rangle = \theta(j - k)$, $j, k \in \mathbb{Z}$. Then $\tilde{T}^* \sum_{k \in \mathbb{Z}} c_k f_k = \sum_{k \in \mathbb{Z}} c_k f_{k-1}$ for all $\{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$. In particular, \tilde{T} is unitary.*

Proof. Consider arbitrary $j, k \in \mathbb{Z}$. Then

$$\langle \tilde{T}f_j, f_k \rangle = \langle f_{j+1}, f_k \rangle = \theta(j + 1 - k) = \langle f_j, f_{k-1} \rangle = \langle f_j, \tilde{T}^* f_k \rangle.$$

It follows that $\tilde{T}^* f_k = f_{k-1}$. Therefore $\tilde{T}\tilde{T}^* = \tilde{T}^*\tilde{T} = I$, i.e., \tilde{T} is unitary and

$$\tilde{T}^* \sum_{k \in \mathbb{Z}} c_k f_k = \sum_{k \in \mathbb{Z}} c_k f_{k-1}$$

for all $\{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$. □

4 Stability of the representation $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$

For applications of frames it is important that key properties are kept under perturbations. We will now state a perturbation condition that preserves the existence of a representation $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$. The condition was first used in connection with frames in the paper [7].

Proposition 4.1 *Assume that $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ is a frame for \mathcal{H} and let $\{g_k\}_{k \in \mathbb{Z}}$ be a sequence in \mathcal{H} . Assume that there exist constants $\lambda_1, \lambda_2 \in [0, 1[$ such that*

$$\left\| \sum c_k (f_k - g_k) \right\| \leq \lambda_1 \left\| \sum c_k f_k \right\| + \lambda_2 \left\| \sum c_k g_k \right\| \quad (4.1)$$

for all finite sequences $\{c_k\}$. Then $\{g_k\}_{k \in \mathbb{Z}}$ is a frame for \mathcal{H} ; furthermore $\{g_k\}_{k \in \mathbb{Z}}$ can be represented as $\{g_k\}_{k \in \mathbb{Z}} = \{V^k g_0\}_{k \in \mathbb{Z}}$ for a linear operator

$$V : \text{span}\{g_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{g_k\}_{k \in \mathbb{Z}}.$$

If T is bounded, then V is also bounded.

Proof. By Theorem 2 in [7] the perturbation condition implies that $\{g_k\}_{k \in \mathbb{Z}}$ is a frame. Also, since $\max(\lambda_1, \lambda_2) < 1$, it follows from (4.1) that

$$\sum_{k \in \mathbb{Z}} c_k f_k = 0 \quad \Leftrightarrow \quad \sum_{k \in \mathbb{Z}} c_k g_k = 0, \quad \forall \{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}). \quad (4.2)$$

Since $\{f_k\}_{k \in \mathbb{Z}}$ is linearly independent, (4.2) implies that the sequence $\{g_k\}_{k \in \mathbb{Z}}$ also is linear independent. Therefore by Proposition 2.1, there is a linear operator $V : \text{span}\{g_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{g_k\}_{k \in \mathbb{Z}}$ such that $\{g_k\}_{k \in \mathbb{Z}} = \{V^k g_0\}_{k \in \mathbb{Z}}$. Now assume that the operator T is bounded; We want to show that then V is also bounded. Let $W : \ell^2(\mathbb{Z}) \rightarrow \mathcal{H}$ be the synthesis operator for $\{g_k\}_{k \in \mathbb{Z}}$, and consider some $\{c_k\}_{k \in \mathbb{Z}} \in N_W$. Then by (4.2), $\{c_k\}_{k \in \mathbb{Z}} \in N_U$, where U is

the synthesis operator for $\{f_k\}_{k \in \mathbb{Z}}$. Since T is bounded, Theorem 2.3 implies that N_U is invariant under right-shifts, i.e., $\sum_{k \in \mathbb{Z}} c_{k-1} f_k = 0$. Using again (4.2), we conclude that $\sum_{k \in \mathbb{Z}} c_{k-1} g_k = 0$, which shows that $\mathcal{T}\{c_k\}_{k \in \mathbb{Z}} \in N_W$. Applying Theorem 2.3 again shows that V is bounded. \square

Note that (4.1) is a special case of the perturbation condition

$$\left\| \sum c_k (f_k - g_k) \right\| \leq \lambda_1 \left\| \sum c_k f_k \right\| + \lambda_2 \left\| \sum c_k g_k \right\| + \mu \left(\sum |c_k|^2 \right)^{1/2}, \quad (4.3)$$

appearing in [7]. If $\{f_k\}_{k \in \mathbb{Z}}$ is a frame for \mathcal{H} with lower bound A , $\{g_k\}_{k \in \mathbb{Z}} \subset \mathcal{H}$, and (4.3) holds for all finite sequences $\{c_k\}$ and some parameters $\lambda_1, \lambda_2, \mu \geq 0$ such that $\max \left(\lambda_2, \lambda_1 + \frac{\mu}{\sqrt{A}} \right) < 1$, then by [7] also $\{g_k\}_{k \in \mathbb{Z}}$ is a frame for \mathcal{H} . This perturbation condition has been used in many different contexts in frame theory, typically for the case $\mu > 0$. However, the case $\mu > 0$ turns out to be problematic if we want the perturbation $\{g_k\}_{k \in \mathbb{Z}}$ of a frame $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ to be represented on the form $\{g_k\}_{k \in \mathbb{Z}} = \{W^k g_0\}_{k \in \mathbb{Z}}$. The first obstacle is that if $\mu > 0$, the perturbation condition (4.3) does not preserve the property of being representable by an operator:

Example 4.2 Consider an orthonormal basis $\{e_k\}_{k \in \mathbb{Z}}$ for a Hilbert space \mathcal{H} . Then the family $\{f_k\}_{k \in I} := \{e_k\}_{k \in \mathbb{Z}} \cup \{\alpha \sum_{j=1}^{\infty} \frac{1}{2^j} e_j\}$ is a linearly independent frame for any choice of $\alpha > 0$, with lower frame bound $A = 1$. For $\alpha < 1$, the family $\{g_k\}_{k \in I} := \{e_k\}_{k \in \mathbb{Z}} \cup \{0\}$ is a perturbation of $\{f_k\}_{k \in I}$ in the sense of (4.3), with $\lambda_1 = \lambda_2 = 0$ and $\mu = \alpha$. However, regardless how small we choose α , the family $\{g_k\}_{k \in I}$ is not linearly independent. Hence, by Proposition 2.1 the sequence $\{g_k\}_{k \in I}$ can not be represented on the form $\{W^k \varphi\}_{k \in \mathbb{Z}}$. \square

The following example shows that even if we assume that the perturbation $\{g_k\}_{k \in \mathbb{Z}}$ of a frame $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ is linearly independent (and hence representable on the form $\{g_k\}_{k \in \mathbb{Z}} = \{V^k g_0\}_{k \in \mathbb{Z}}$), the condition (4.3) does not imply that V is bounded if T is bounded.

Example 4.3 Let us first explain the idea of the construction in the setting of a general Hilbert space \mathcal{H} . Assume that $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ is an overcomplete frame for \mathcal{H} , with lower bound A , and that the operator T is bounded. We further assume that

- (a) The sequence $\{f_k\}_{k \in \mathbb{Z} \setminus \{-1, 0\}}$ is complete in \mathcal{H} .

We will then search for some $g_0 \in \mathcal{H}$ such that the sequence

$$\{g_k\}_{k \in \mathbb{Z}} := \{f_k\}_{k=-\infty}^{-1} \cup \{g_0\} \cup \{f_k\}_{k=1}^{\infty}$$

satisfy the following requirements:

- (b) The condition (4.3) is satisfied with $\max\left(\lambda_2, \lambda_1 + \frac{\mu}{\sqrt{A}}\right) < 1$;
- (c) $\{g_k\}_{k \in \mathbb{Z}}$ is linearly independent.

We will now explain how this setup leads to the desired conclusion; after that we provide a concrete construction satisfying all the requirements.

First, the condition (b) implies that $\{g_k\}_{k=1}^{\infty}$ is a frame for \mathcal{H} ; by (c) it has the form $\{g_k\}_{k \in \mathbb{Z}} = \{W^k g_0\}_{k \in \mathbb{Z}}$ for some operator

$$W : \text{span}\{g_k\}_{k \in \mathbb{Z}} \rightarrow \text{span}\{g_k\}_{k \in \mathbb{Z}}.$$

By the definition of the sequence $\{g_k\}_{k \in \mathbb{Z}}$ it follows that

$$\begin{cases} Wf_k &= f_{k+1}, \quad k = -2, -3, \dots \\ Wf_{-1} &= g_0, \\ Wg_0 &= f_1, \\ Wf_k &= f_{k+1}, \quad k = 1, 2, \dots \end{cases} \quad (4.4)$$

We note that the operators T and W act in an identical way on the vectors $\{f_k\}_{k \in \mathbb{Z} \setminus \{-1, 0\}}$; thus, if W is bounded it follows by (a) that $W = T$. But then (4.4) implies that $g_0 = Wf_{-1} = Tf_{-1} = f_0$, i.e., that $\{f_k\}_{k \in \mathbb{Z}} = \{g_k\}_{k \in \mathbb{Z}}$. In other words: for a perturbation satisfying the stated conditions, the operator W in the representation $\{g_k\}_{k \in \mathbb{Z}} = \{W^k g_0\}_{k \in \mathbb{Z}}$ will not be bounded when $g_0 \neq f_0$.

We now proceed to a concrete construction satisfying (a)–(c). In order to do so, we return to the shift-invariant systems considered in Example 1.1. First, it is well-known that the function $\text{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$ generates an orthonormal basis $\{T_k \text{sinc}\}_{k \in \mathbb{Z}}$ for the Paley-Wiener space

$$\mathcal{H} := \{f \in L^2(\mathbb{R}) \mid \text{supp } \hat{f} \subseteq [-1/2, 1/2]\}.$$

It follows that the oversampled family $\{f_k\}_{k \in \mathbb{Z}} := \{T_{k/3} \text{sinc}\}_{k \in \mathbb{Z}} = \{T_{1/3}^k \text{sinc}\}_{k \in \mathbb{Z}}$ can be considered as a union of three orthonormal bases, and hence form a

tight frame for \mathcal{H} ; we note that by the carefully chosen oversampling, the condition (a) is satisfied. The operator $T := T_{1/3}$ is clearly bounded.

Now, consider a constant $c \geq 0$ and let $g_0 := T_c f_0$; then $g_0 \in \mathcal{H}$, and for any finite scalar sequence $\{c_k\}$ we have

$$\left\| \sum c_k (f_k - g_k) \right\| = \|c_0(f_0 - T_c f_0)\| \leq \|f_0 - T_c f_0\| \left(\sum |c_k|^2 \right)^{1/2}.$$

By continuity of the translation operator there exists some $\delta > 0$ such that $\|f_0 - T_c f_0\| < \sqrt{A}$ whenever $c \in [0, \delta[$; it now follows from the perturbation condition that $\{g_k\}_{k \in \mathbb{Z}}$ is a frame for \mathcal{H} for c belonging to this range, i.e., the condition (b) is satisfied. Furthermore, for $c < 1/3$ all the translation parameters appearing in the sequence $\{g_k\}_{k \in \mathbb{Z}}$ are pairwise different; thus $\{g_k\}_{k \in \mathbb{Z}}$ is linearly independent and condition (c) is fulfilled. \square

Most of the concrete applications of perturbation results in frame theory deals with the special case of the condition (4.3) corresponding to $\lambda_1 = \lambda_2 = 0$. Even in this case, Example 4.3 shows that the perturbation condition does not preserve boundedness of the representing operator. In applications where stability is an important issue, one can alternatively represent a frame using iterated operator systems based on a finite collection of operators instead of a singleton. Consider a frame $\{f_k\}_{k \in \mathbb{Z}}$ which is norm-bounded below. It is proved in [10] that then there is a finite collection of vectors from $\{f_k\}_{k \in \mathbb{Z}}$, to be called $\varphi_1, \dots, \varphi_J$, and corresponding bounded operators $T_j : \mathcal{H} \rightarrow \mathcal{H}$, such that $\{T_j^n \varphi_j\}_{n \in \mathbb{Z}}$ is a Riesz sequence, and $\{f_k\}_{k \in \mathbb{Z}} = \cup_{j=1}^J \{T_j^n \varphi_j\}_{n \in \mathbb{Z}}$. The proof uses the Feichtinger theorem (which was a conjecture for several years and finally got confirmed in [15]). We will now show that the stated representation is stable with respect to the central perturbation condition (4.3) with $\lambda_1 = \lambda_2 = 0$.

Theorem 4.4 *Assume that the frame $\{f_k\}_{k \in \mathbb{Z}}$ is norm-bounded below, and consider a representation on the form $\cup_{j=1}^J \{T_j^n \varphi_j\}_{n \in \mathbb{Z}}$, where the operators T_j are bounded and $\{T_j^n \varphi_j\}_{n \in \mathbb{Z}}$ is a Riesz sequence. Let A denote a common lower frame bound for the frame $\{f_k\}_{k=1}^\infty$ and all the Riesz sequences $\{T_j^n \varphi_j\}_{n \in \mathbb{Z}}$, $j = 1, \dots, J$. Let $\{g_k\}_{k \in \mathbb{Z}}$ be a sequence in \mathcal{H} such that for some $\mu < \sqrt{A}$,*

$$\left\| \sum c_k (f_k - g_k) \right\| \leq \mu \left(\sum |c_k|^2 \right)^{1/2}, \quad (4.5)$$

for all finite scalar sequences $\{c_k\}$. Then there is a finite collection of vectors ϕ_1, \dots, ϕ_J from $\{g_k\}_{k \in \mathbb{Z}}$ and corresponding bounded operators $W_j : \mathcal{H} \rightarrow \mathcal{H}$, such that $\{W_j^n \phi_j\}_{n \in \mathbb{Z}}$ is a Riesz sequence, and

$$\{g_k\}_{k \in \mathbb{Z}} = \bigcup_{j=1}^J \{W_j^n \phi_j\}_{n \in \mathbb{Z}}. \quad (4.6)$$

Proof. The perturbation condition (4.5) is a special case of (4.3); thus the family $\{g_k\}_{k=1}^\infty$ is a frame for \mathcal{H} . Furthermore, partitioning $\{g_k\}_{k \in \mathbb{Z}}$ according to the splitting $\{f_k\}_{k \in \mathbb{Z}} = \bigcup_{j=1}^J \{T_j^n \varphi_j\}_{n \in \mathbb{Z}}$, i.e., writing $\{g_k\}_{k \in \mathbb{Z}} = \bigcup_{j=1}^J \{g_j^{(n)}\}_{n \in \mathbb{Z}}$, it follows from (4.5) that for any fixed $j \in \{1, \dots, J\}$ and any finite scalar sequence $\{c_n\}_{n \in \mathbb{Z}}$,

$$\left\| \sum c_n (T_j^n \varphi_j - g_j^{(n)}) \right\| \leq \mu (\sum |c_n|^2)^{1/2}.$$

Therefore, for each fixed $j \in \{1, \dots, J\}$ the sequence $\{g_j^{(n)}\}_{n \in \mathbb{Z}}$ is a Riesz sequence, and hence representable on the form $\{W_j^n \phi_j\}_{n \in \mathbb{Z}}$ for some bounded operator W_j and some $\phi_j \in \mathcal{H}$.

Note that an alternative way of proving the result would be to show directly that (4.5) implies that $\{g_k\}_{k \in \mathbb{Z}}$ is norm-bounded below and then refer to the stated result in [10]. However, this argument would not yield that the splitting of the indexing of the frame $\{f_k\}_{k \in \mathbb{Z}}$ is preserved for the perturbed family $\{g_k\}_{k \in \mathbb{Z}}$, as in (4.6). \square

Appendix: auxiliary examples

We will close the paper with a few operator-theoretical considerations, to which we have referred throughout the paper.

1) Instead of representing a frame on the form $\{T^k f_0\}_{k \in \mathbb{Z}}$, one could also consider representations on the form $\{T^n f_0\}_{n=0}^\infty$; this indexing occurs, e.g., in dynamical sampling [2, 3]. The chosen indexing actually has a serious influence on the properties of the operator T . For example, there exist frames $\{f_k\}_{k \in \mathbb{Z}} = \{T^k f_0\}_{k \in \mathbb{Z}}$ where T is a unitary operator (Example 1.1); but if we reindex the frame as $\{f_k\}_{k=0}^\infty$ it can not be represented on the form $\{U^n f_0\}_{n=0}^\infty$ for a unitary operator U , see [3].

Let us demonstrate the sensibility to the indexing by one more case. First, it is well-known that the function $\text{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$ generates an orthonormal basis $\{T_k \text{sinc}\}_{k \in \mathbb{Z}}$ for the Paley-Wiener space

$$\mathcal{H} := \{f \in L^2(\mathbb{R}) \mid \text{supp } \hat{f} \subseteq [-1/2, 1/2]\}.$$

It follows that the oversampled family $\{T_{k/2} \text{sinc}\}_{k \in \mathbb{Z}} = \{T_{1/2}^k \text{sinc}\}_{k \in \mathbb{Z}}$ is a tight frame for \mathcal{H} ; the representing operator $T_{1/2}$ is clearly bounded.

On the other hand, the following lemma shows that considering $\{T_{k/2} \text{sinc}\}_{k \in \mathbb{Z}}$ as a union of the two orthonormal bases $\{T_k \text{sinc}\}_{k \in \mathbb{Z}}$ and $\{T_{1/2} T_k \text{sinc}\}_{k \in \mathbb{Z}}$, re-indexing in a natural fashion as $\{T^n f_0\}_{n=0}^\infty$ always leads to an unbounded operator T .

Lemma 4.5 *Consider two orthonormal bases $\{f_k\}_{k=1}^\infty$ and $\{e_k\}_{k=1}^\infty$ for a Hilbert space \mathcal{H} and assume that the set $\{f_k\}_{k=1}^\infty \cup \{e_k\}_{k=1}^\infty$ is linearly independent. Then a linear operator $T : \text{span}(\{f_k\}_{k=1}^\infty \cup \{e_k\}_{k=1}^\infty) \rightarrow \mathcal{H}$ such that*

$$\{\varphi_k\}_{k=0}^\infty := \{f_1, e_1, f_2, e_2, \dots\} = \{T^k \varphi_0\}_{k=0}^\infty, \quad (4.7)$$

is necessarily unbounded.

Proof. The ordering in (4.7) implies that $T f_k = e_k$ for all $k \in \mathbb{N}$; thus, if the operator T is bounded, it has a unique extension to a bounded linear operator $\tilde{T} : \mathcal{H} \rightarrow \mathcal{H}$, given by $\tilde{T} \sum_{k=1}^\infty c_k f_k = \sum_{k=1}^\infty c_k e_k$, $\{c_k\}_{k=1}^\infty \in \ell^2(\mathbb{N})$. Clearly \tilde{T} is a surjective mapping. On the other hand, (4.7) also implies that $\tilde{T} e_k = f_{k+1}$; since $\{e_k\}_{k=1}^\infty$ is an orthonormal basis for \mathcal{H} this implies that the range of T equals the space $\overline{\text{span}}\{f_k\}_{k=2}^\infty$, which excludes that \tilde{T} is surjective. This contradiction shows that the operator T can not be bounded. \square

2) Let V denote a dense subspace of a Hilbert space \mathcal{H} , and consider a bounded and bijective operator $T : V \rightarrow V$. Then T has a unique extension to a bounded operator $\tilde{T} : \mathcal{H} \rightarrow \mathcal{H}$. The following example demonstrates that the extension \tilde{T} might no longer be injective.

Example 4.6 Let $\{e_k\}_{k=1}^\infty$ denote an orthonormal basis for a separable Hilbert space, and consider the sequence $\{f_k\}_{k=1}^\infty := \{e_1\} \cup \{e_{k-1} + \frac{1}{k} e_k\}_{k=2}^\infty$. Then $\{f_k\}_{k=1}^\infty$ is a frame (see [9]), and it is easy to see that the elements are linearly independent. Let $V := \text{span}\{f_k\}_{k=1}^\infty$, and consider the operator

$T : V \rightarrow V$, $Tf := \sum_{k=1}^{\infty} \langle f, e_k \rangle f_k$. Then T is linear and bounded, its extension to a bounded operator on \mathcal{H} is given by the same expression, i.e., $\tilde{T} : \mathcal{H} \rightarrow \mathcal{H}$, $\tilde{T}f = \sum_{k=1}^{\infty} \langle f, e_k \rangle f_k$. The operator $T : V \rightarrow V$ is bijective. Injectivity follows from the fact that for $f \in V$, the sequence $\{\langle f, e_k \rangle\}_{k=1}^{\infty}$ is finite; hence, due to the linear independence of $\{f_k\}_{k=1}^{\infty}$ we can only have $Tf = 0$ if $\langle f, e_k \rangle = 0$ for all $k \in \mathbb{N}$, which implies that $f = 0$. So show that T is surjective, let $g \in V$. Then $g = \sum_{k=1}^N c_k f_k$ for some $N \in \mathbb{N}$ and some $c_k \in \mathbb{C}$. Then $f := \sum_{k=1}^N c_k e_k \in V$ and $Tf = g$.

However, the bounded extension $\tilde{T} : \mathcal{H} \rightarrow \mathcal{H}$ is not injective. Indeed, since $\{f_k\}_{k=1}^{\infty}$ is an overcomplete frame, there exist coefficients $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N}) \setminus \{0\}$ such that $\sum_{k=1}^{\infty} c_k f_k = 0$; taking $f := \sum_{k=1}^{\infty} c_k e_k \neq 0$ we have $Tf = 0$. \square

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